Game Theory and Linear Algebra

Erin Tripp (Syracuse University)

August 7, 2015

Mentor: Bruce Suter (AFRL)
Collaborators: Lixin Shen (SU), Jérôme Bolte (Toulouse 1 University Capitole)
Overview

1. Game theory
   ▶ Definitions and Notation
   ▶ Solution Concepts
   ▶ Finding Solutions

2. Application to Linear Algebra
   ▶ Idea
   ▶ Example problem
   ▶ Solution methods
Game Theory basics

Game theory is the study of strategic decision making and interaction. The goal is to predict behavior and therefore predict outcomes. It has applications to a wide variety of fields, such as political science, economics, and computer science, but it is also a well established mathematical theory.
To specify a game, we need the following (PAPI):

- The number of **players**
- The **actions** available to those players
- Their **preferences** over these actions/outcomes
- The **information** available to each player
Definitions

Formally, a game in normal form is a tuple $G = (N, A, u)$ where:

- $N = \{1, 2, ..., n\}$ is the set of players
- $A = \prod_{i \in N} A_i$ is the set of action profiles, where $A_i$ are the actions available to player $i$
- $u = \prod_{i \in N} u_i : A \rightarrow \mathbb{R}$ is the utility function, which characterizes player's preferences over the actions. Each player acts to maximize their utility.

A **strategy** for player $i$ $s_i$ is a probability distribution over their actions $A_i$. Notation:

- $s_i(a_i)$ – the probability of $a_i$ under $s_i$
- $u_i(s)$ – the expected utility for player $i$ under strategy $s$

If the support of this distribution is a single action, it is called a pure strategy. Otherwise, it is a mixed strategy. A strategy profile, like an action profile, specifies a strategy for each player.
Example

Matching Pennies.
In this game, each player puts down a penny. If the coins match, player 1 takes both for a total gain of one cent. If the coins do not match, player 2 takes both. This is a two player, zero sum game with two actions available to each player, represented in normal form below.

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>T</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>
Zero sum games

These are games of pure competition in which players’ interests are
diametrically opposed. Zero sum games are most meaningful for two
players.

A game is zero sum if $\forall a \in A: u_1(a) + u_2(a) = 0$

More generally, a competitive game is constant sum if $\forall a \in A$:
$u_1(a) + u_2(a) = c$. 
Continuous games

Traditionally, game theory dealt with a finite number of players each with a finite number of actions. However, for many applications, this may be too restrictive. A game is called continuous if the action space $A$ is compact and the utility function $u$ is continuous. We still assume finitely many players, but we now have infinitely many pure strategies.

As you may have guessed, we are interested in continuous, zero sum, two player games.
Solution Concepts

Solution concepts are formal predictions of what strategies players will use in particular circumstances. We are interested in two such concepts:

- Nash equilibrium
- Minimax strategies

In a Nash equilibrium, no player stands to improve their utility by changing strategies if all other player’s strategies are fixed.

Closely related to Nash equilibria are epsilon equilibria. In an $\epsilon$-equilibrium, players may stand to gain some small utility by deviating from the equilibrium strategy, but this gain is bounded above by $\epsilon$.

A minimax strategy for a player minimizes their worst case loss. This is particularly useful for zero sum games.
Some important results

Theorem 1 (Nash 1951)
Every game with a finite number of players and action profiles has a mixed strategy Nash equilibrium.

Theorem 2 (von Neumann 1928)
In any two player, zero sum game with finitely many action profiles, every Nash equilibrium is a minimax strategy. (paraphrased)

Lemma 3
Let G be a finite game in normal form. Then $s$ is a mixed strategy Nash Equilibrium of G if and only if every pure strategy in the support of $s_i$ is a best response to $s_{-i}$ for all $i \in N$. ¹

¹From Osborne and Rubinstein’s A Course in Game Theory
Even more important results \(^2\)

**Theorem 4 (Glicksberg 1952)**

*Every continuous game has a mixed strategy Nash equilibrium.*

**Proposition 1**

*Assume that* \(s^k \to s\), \(\epsilon^k \to \epsilon\), *and for each* \(k\) *\(s^k\) is an* \(\epsilon^k\)-*equilibrium of* \(G\). *Then* \(s\) *is an* \(\epsilon\)-*equilibrium of* \(G\).*

Suppose \(G\) and \(G'\) are games which differ only by their utility functions \(u\) and \(u'\) respectively. Then we say \(G'\) is an \(\alpha\)-approximation of \(G\) if

\[ |u_i(a) - u'_i(a)| \leq \alpha \quad \forall i \in N \text{ and } a \in A. \]

**Proposition 2**

*For any continuous game* \(G\) *and any* \(\alpha > 0\), *there exists an “essentially finite” game which is an* \(\alpha\)-*approximation of* \(G\).*

---

\(^2\)Asu Ozdaglar. 6.254 Game Theory with Engineering Applications, Spring 2010. (Massachusetts Institute of Technology: MIT OpenCourseWare)
Finding equilibria in finite games

To find a pure strategy equilibrium in a finite game, iterative methods such as removal of strictly dominated strategies reduce it to a smaller game with the same equilibria but which is much easier to sort through.

There is no guarantee that a pure strategy Nash equilibrium exists.

Finding mixed equilibria, even in finite games, is not an easy problem. The standard approach is to guess the support of an equilibrium strategy and set the expected utilities equal.

For two players with only two actions, this is trivial.
Example

\[\begin{array}{c|cc}
 & H & T \\
\hline
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array}\]

Suppose player 1 mixes Heads and Tails with probabilities \(p\) and \(1 - p\) respectively.

By Lemma 3, this mixing must make player 2 indifferent between Heads and Tails.

\[u_2((p, 1-p), H) = u_2((p, 1-p), T)\]
\[-1 \cdot p + 1 \cdot (1 - p) = 1 \cdot p - 1 \cdot (1 - p)\]
\[p = \frac{1}{2}, 1 - p = \frac{1}{2}\]
Finding equilibria in continuous games

Equilibria are fixed points of a set-valued best response function, which takes strategies for each player to strategies which are better responses (regarding the others as fixed). Fixed point theorems are generally used to prove the existence of equilibria for particular classes of games. A constructive method along with the appropriate best response function could locate an equilibrium.

A more computational approach is to take successively finer discretizations of the game in question and find a sequence of convergent pure strategy equilibria which will converge to a mixed strategy equilibrium in the original game.
Application to Linear Algebra

Inspired by CalTech Professor Houman Owhadi’s work on PDE’s: Owhadi characterized the process of solving a PDE as a zero sum game of incomplete information. He was able to find a fast solver for a particular class of PDE’s in this way.

Similarly, we want to frame general problems in linear algebra as zero sum games and find optimal methods of solving them.
Bolte’s game

We considered a two player, zero sum game, in which player 1 chooses a problem, and player 2 chooses an algorithm to solve the problem.

In particular, we looked at problems of the form $Ax = 0$, where 1 chose $A$ to maximize the error in 2’s solution. Player 2 applied the gradient descent algorithm, choosing the step sizes as well as the number of steps.

Player 2 wants an optimal strategy to this game, which will guarantee the smallest error in the solution given that Player 1 wants to do the most harm.

We assume that $\|A^T A\| \leq L$ and that possible step sizes are bounded by some $M \in \mathbb{R}$.

(With these restrictions, the strategy space is compact.)
Bolte’s game

If $x_k$ is player 2’s solution, their loss is given by:

$$\text{err}_{x_k} = \|Ax_k - 0\|^2_2 - \min(f) = \|Ax_k\|^2_2 = \langle A^T Ax_k, x_k \rangle$$  \hspace{1cm} (1)

$A^T A$ symmetric, real $\rightarrow A^T A = \Omega^T \Delta \Omega$, where $\Omega$ orthogonal and $\Delta$ diagonal. Substituting this product into (1) and manipulating gives:

$$\text{err}_{x_k} = \frac{\|x_0\|}{2} |\delta(1 - \lambda_0 \delta) \cdots (1 - \lambda_k \delta)|$$  \hspace{1cm} (2)

where $x_0$ is the initial guess, $\delta$ is the largest eigenvalue of $A^T A$, and $\lambda_i$ is the size of the $i$-th step.

(This function is continuous, so we now have a continuous game!)
Bolte’s game

Player 1
- $Ax = 0$, where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $\|ATA\| \leq L \in \mathbb{R}$.
- Find $x^* = \text{arg}(\min_x \frac{1}{2}\|Ax\|^2)$.

Player 2
- Fix $x_0 \in \mathbb{R}^n$. For $i = 0, \ldots, k - 1$, let $\lambda_i \in \mathbb{R}$ and set $x_{i+1} = x_i - \lambda_i \nabla f(x_i)$.
- Solution = $x_k$,
  $$err_{x_k}(\delta, \lambda) = \frac{1}{2}\|Ax_k\|^2 = \frac{||x_0||^2}{2}\delta(1 - \lambda_k \delta)...(1 - \lambda_0 \delta)$$

An equilibrium strategy $(\delta, \lambda_0, \ldots, \lambda_k)$ for this game must satisfy the following:

$$\min_{\lambda} \max_{\delta} err_{x_k}(\delta, \lambda) = \max_{\delta} \min_{\lambda} err_{x_k}(\delta, \lambda) \quad (3)$$
Solution methods

We considered two methods of solving this game. Both involve making the game finite.

- A single, finite game, for which we compute a mixed strategy equilibrium as before.
- A sequence of finite games, for which we compute pure strategy equilibria.

Furthermore, we separated the game into cases:

- Fixed: \( \lambda_i = \lambda \in \mathbb{R} \ \forall i \)
- Alternating: \( \lambda_i = \lambda_0 \) or \( \lambda_1 \ \forall i \)
- General: Possibly all \( \lambda_i \) distinct.
Single finite game

Very simplistic method:

- Let $-L, L$ and $-M, M$ be the pure strategies.
- Compute mixed strategy equilibrium as for Matching Pennies
- Mixed strategy equilibrium is a convex combination of pure strategies (i.e. a point in the intervals $[-L, L], [-M, M]$)

Results:

- Fixed: choose step size $= 0$
- Alternating: choose same step size in opposite directions
- General: not enough information!

This approach is quick and simple, but does not produce exact solutions and can not be generalized.
Sequence of finite games

Process:
1. Divide strategy intervals with step size \( n \)
2. Find minmax strategies for this level
3. Repeat with smaller step size
4. Find convergent sequence of equilibrium as \( n \to 0 \)

\[
\begin{array}{c|cc}
 & -M & M \\
\hline
-L & \text{err}(-L, -M) & \text{err}(-L, M) \\
L & \text{err}(L, -M) & \text{err}(L, M) \\
\end{array}
\]

\[
\begin{array}{c|ccc}
 & -M & 0 & M \\
\hline
-L & \text{err}(-L, -M) & \text{err}(-L, 0) & \text{err}(-L, M) \\
0 & \text{err}(0, -M) & \text{err}(0, 0) & \text{err}(0, M) \\
L & \text{err}(L, -M) & \text{err}(L, 0) & \text{err}(L, M) \\
\end{array}
\]
Currently using Python to process this for various values of $k$.

Notes:

- Error values explode away from zero, so we use fairly small intervals
- This method produces several equilibria strategies at each stage

For example,
(Fixed) if $L = 4$, $M = 4$, $k = 2$, an approximated equilibrium strategy is $(0.082, 4.0)$. This gives an error of 0.037.
Future plans

We hope to develop a smarter program which does not exhaustively search through every possible strategy and to find solutions for the alternating and general cases.

We will also explore how the search for optimal strategies changes when some additional structure is imposed on the matrix $A$ (e.g. banded, hierarchical, etc.) and if the game itself can produce an algorithm or improve existing algorithms.
Thank you!
References


